Summary of Physics Topics Using Fourier Series

P111/141

Sound waves and beats

Standing waves

Fourier analysis: Interference

P243

Wave packet, superposition of waves

Fourier transform: uncertainty principle

P250

Fourier series and analysis for periodic function

Driven oscillator

P371

Solution to Laplace equation with boundary conditions

Use orthogonality to project and find the coefficients

Fourier Series in Physics

1. Addition of Sinusoidal Functions: Superposition Principle in Physics

When two or more waves traverse the same region, they act independently of each other. According to the principle of superposition, we add the displacements of all waves present at a certain time. The net displacement depends on the harmonic amplitude, the phase and the frequency of each of the individual waves. This can lead to constructive and destructive interference effects. The superposition principle also allows us to build complex wave shapes by superposing simpler ones.

1.1 Two sound waves of nearly equal frequencies f_1 and f_2

Beats are observed.

Pluck two guitar strings that have slightly different frequencies, and you will notice that the sound produced by the strings is not constant in time. Instead, the intensity increases and decreases with a definite period of time. These fluctuations in intensity are the beats, and the frequency of successive maximum intensities is the beat frequency.

Assume the two waves can be described as

$$y_1(t) = A cos(2\pi f_1 t)$$

$$y_2(t) = A cos(2\pi f_2 t)$$

Adding them together, the total wave takes a form of

$$y_{total}(t) = y_1(t) + y_2(t) = A cos\left(2\pi f_1 t\right) + A cos\left(2\pi f_2 t\right) = 2A cos(2\pi \frac{f_1 - f_2}{2} t) cos(2\pi \frac{f_1 + f_2}{2} t)$$

The first term represents the slowly varying amplitude of the beats giving the beat frequency

$$f_{beat} = \left| f_1 - f_2 \right|$$

The rapid oscillation within each beat are due to the second part, where the frequency takes an average of the two individual frequencies f_1 and f_2.

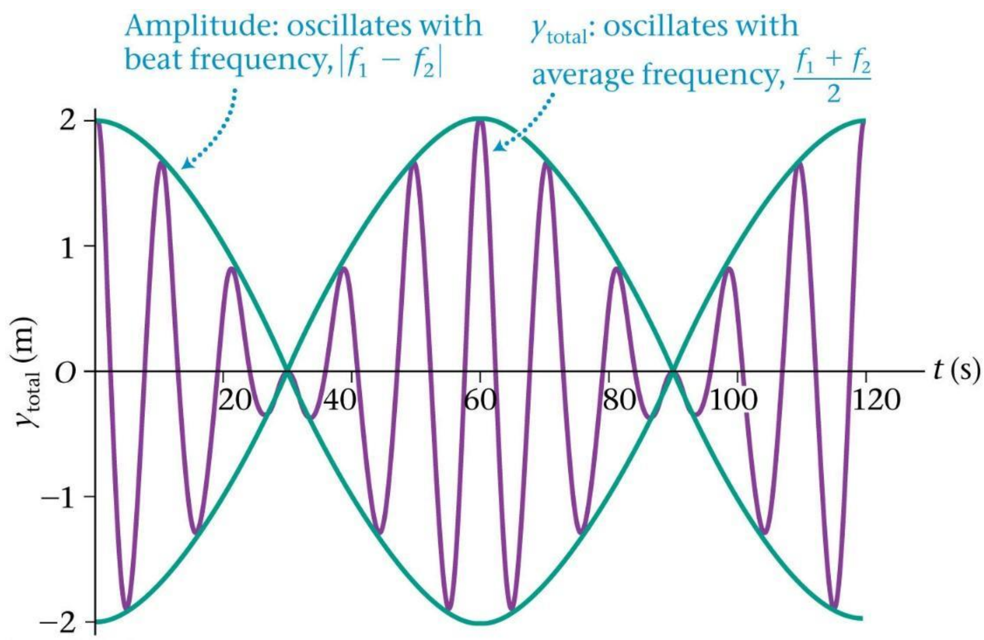


Figure 1. Superposition of two sound waves to produce the beats. (Image is cited from Physics by James Walker.)

1.2 Standing waves

The formation of standing waves is also due to the superposition of two traveling waves- the incident and the reflected waves propagating in the opposite directions. In this case, the two waves approaching from opposite directions. At certain locations, the wave crests coincide and so do the troughs. The resulting waves is, momentarily twice as big. This is constructive interference- two waves superposing to produce a larger wave displacement, called the antinodes. At some other locations, the two waves cancel, this is destructive interference, called the nodes.

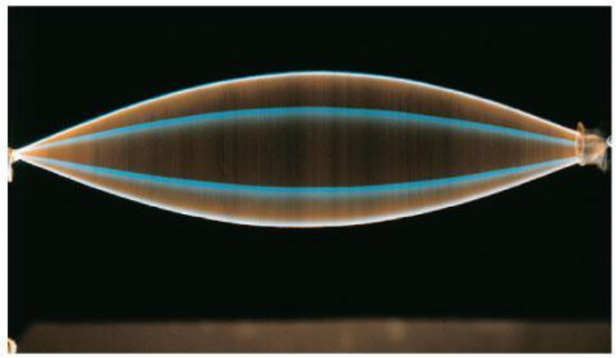
Assume the two traveling waves can be described as

$$y_1(t) = Asin(\frac{2\pi}{\lambda}x - 2\pi ft)$$

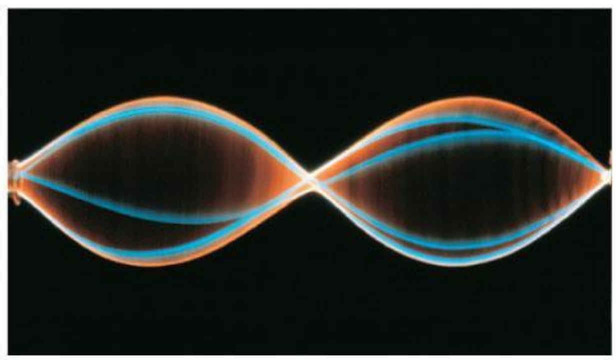
$$y_2(t) = Asin(\frac{2\pi}{\lambda}x + 2\pi ft)$$

Adding them together, the total wave takes a form of

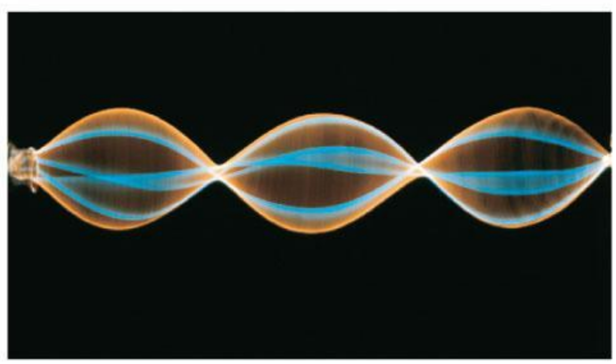
$$y_{total}(t) = y_1(t) + y_2(t) = Asin\left(\frac{2\pi}{\lambda}x - 2\pi ft\right) + Asin(\frac{2\pi}{\lambda}x + 2\pi ft) = 2Asin(\frac{2\pi}{\lambda}x)cos(2\pi ft)$$



(a)



(b)



(c)

Figure 2. Standing wave patterns for (a) the lowest frequency standing wave- also referred to as the fundamental, or first harmonic, which has one antinode; (b)the second harmonic which has two antinodes; and (c) the third harmonic which has three antinodes. (Image is cited from Physics by James Walker.)

1.3 More examples of superposition of multiple waves

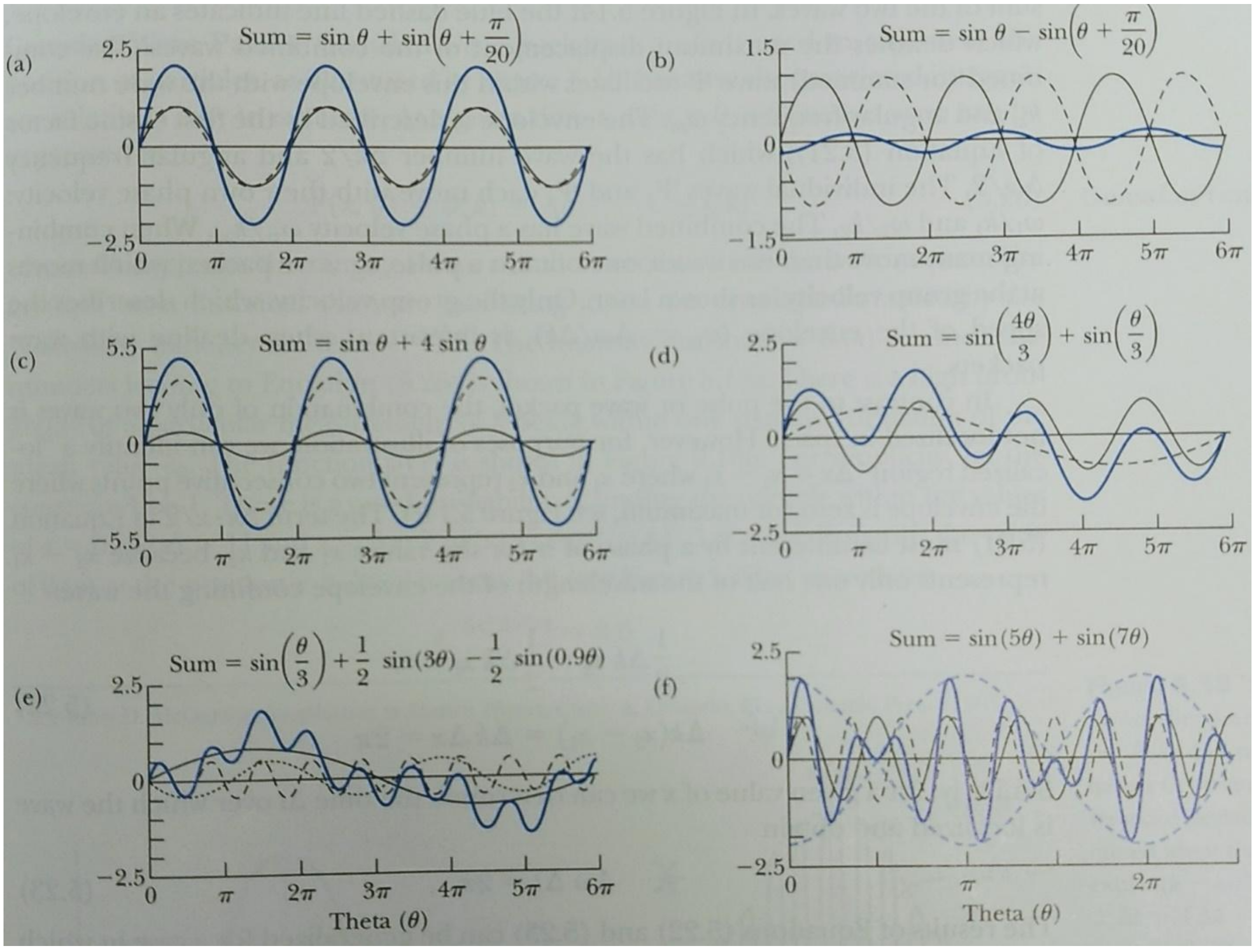


Figure 3. Superposition of Waves. The heavy blue line is the resulting wave. (a) Two waves of equal frequency and amplitude that are almost in phase. The results is a larger wave. (b) Two waves of equal frequency and amplitude that are almost out of phase. The results is a smaller wave. (c) Two waves with the same frequency, but different amplitudes. (d) Two waves of equal amplitude but different frequencies. (e) Three waves of different amplitudes and frequencies. (f) Two waves of close frequencies with the same amplitudes over many wavelengths creating the phenomenon of beats. (Image is cited from Modern Physics For Scientists and Engineers by Stephen T. Thornton and Andrew Rex.)

2. Decomposition of Complex Functions/Waves into Sinusoidal Functions/Waves

The French mathematician Joseph Fourier showed that any periodic wave can be written as a sum of simple sinusoidal (harmonic) waves, a process now known as Fourier analysis. Fourier analysis has applications ranging from music to structural engineering, and to communications, because it helps us understand how a complex wave behaves if we know how its harmonic components behave.

The Fourier's theorem states that every periodic function with a period of T (i.e. frequency f=1/T) can be written as a linear combination of the sines and cosines where the constants a_n and b_n depend on the function $f(t)$.

$$f(t) = \sum_{n=0}^{\infty} [a_n \cos cos\,(2\pi nft) + b_n \sin sin\,(2\pi nft)] = \sum_{n=0}^{\infty} [a_n \cos cos \left(\frac{2\pi nt}{T}\right) + b_n \sin sin \left(\frac{2\pi nt}{T}\right)]$$

The sum is called the Fourier series. This can equally be written down in terms of wavelength in the spatial coordinates. Although this sum includes infinite terms ($n\to\infty$), it is often the case that one gets an excellent approximation by retaining just the first few terms of a Fourier series. Thus, we only have to handle a reasonably small number of sines and cosines. The coefficients can be determined by

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \cos cos \left(\frac{2\pi nt}{T}\right) dt \qquad for\ n\geq 1$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \sin sin \left(\frac{2\pi nt}{T}\right) dt \qquad for\ n\geq 1$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt \quad and \quad b_0 = 0$$

2.1 Decomposing sound waves from an electric guitar

The mix of Fourier components in the sound wave from a musical instrument determines the exact sound we hear and accounts for the different sounds from different instruments even when they are playing the same notes.

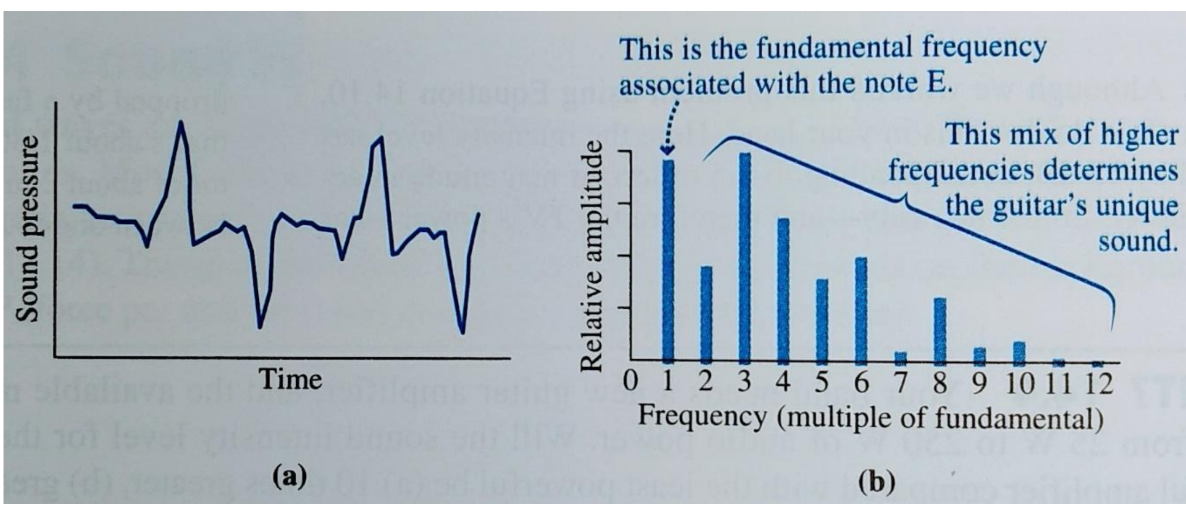


Figure 4. (a) An electric guitar plays the note E, producing a complex wave. (b) Fourier analysis shows the relative strengths of the individual sine waves whose sum produces the total wave. (Image is cited from University Physics by Richard Wolfson.)

2.2 Fourier series for a rectangular pulse

Periodic rectangular pulse *f*(*t*) has a time period of τ, a duration of Δτ and a pulse height of f_{max} (see Figure 5). It can be described as follows:

$$f(t) = \{f_{mzx} \quad when\, m\tau - \frac{\Delta\tau}{2} \le t \le m\tau + \frac{\Delta\tau}{2}\, ,\, m\, is\, any\, integer\, 0$$

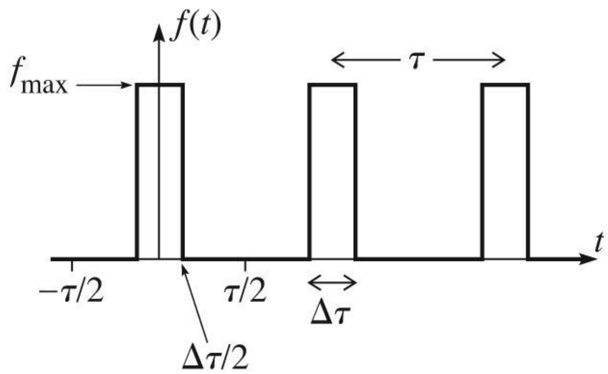


Figure 5. A periodic rectangular pulse. The period is τ, the duration of the pulse is Δτ and the pulse height is f_{max}. (Image is cited from Classical Mechanics by John R. Taylor.)

Find the Fourier coefficients a_n and b_n for this function.

For the cosine terms:

$$a_0 = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} f(t)dt = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} f_{max} dt = \frac{f_{max}\Delta\tau}{\tau}$$

$$a_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} f(t) \cos cos \left(\frac{2\pi n t}{\tau}\right) dt = \frac{2f_{max}}{\pi n} sin\left(\frac{\pi n \Delta\tau}{\tau}\right) \quad for\, n > 0$$

For the sine terms:

$$b_n = 0$$

The Fourier series is therefore

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos cos \left(\frac{2\pi n t}{\tau}\right)$$

Now take some numerical values: $\tau = 1,\ \Delta\tau = 0.25\ and\ f_{max} = 1$

One can calculate all the terms

$$f(t) = f_{max}[0.25 + 0.45 \cos cos\,(2\pi t) + 0.32 \cos cos\,(4\pi t) + 0.15 \cos cos\,(6\pi t) + 0 \cos cos\,(8\pi t)$$

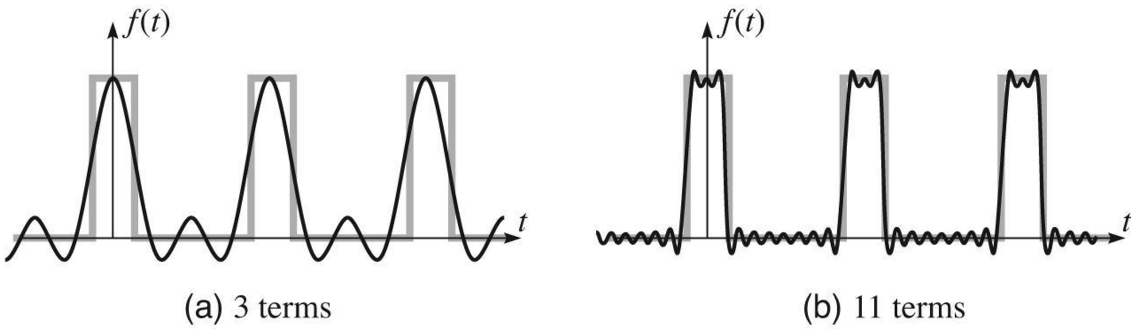


Figure 6. Fourier analysis of a periodic rectangular pulse with (a) the sum of the first 3 terms of the Fourier series and (b) the sum of the first 11 terms of the Fourier series. (Image is cited from Classical Mechanics by John R. Taylor.)

This series will converge as *n* increases. See Figure 6 for an illustration of using the first 3 and 11 terms. As you may expect, with only 3 terms we may capture the overall shape of the curve, but miss largely on an accurate approximation of the discontinuous function. When one includes 11 terms, the fitting is pretty good.