

VERTICAL TRANSMISSION IN EPIDEMIC MODELS OF SEXUALLY TRANSMITTED DISEASES WITH ISOLATION FROM REPRODUCTION.

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ABSTRACT. We describe a population logistic model exposed to a mild life-long sexually transmitted disease, i.e. without significant increased mortality among infected individuals and providing no immunity/recovery. We then modify this model to include groups isolated from sexual contact and analyze their potential effect on the dynamics of the population. We are interested in how the isolated class may curb the growth of the infected group while keeping the healthy population at acceptable levels. In particular, we analyze the connection between vertical transmission and isolation from reproduction on the long term behavior of the disease.

1. INTRODUCTION

The dynamics of a population depend on the relation between reproduction and mortality. One factor that we analyze in this paper is the long-term effect on the population growth caused by the segregation of portions of the general (reproductive) population into a non-reproductive class that really consists of individuals of two very different kinds: *sexually active but non-procreating*, such as infertile individuals, and *sexually inactive*, consisting of individuals who by choice or medical reasons refrain from sexual contact for life. The influence of the non-reproductive group on general population dynamics has been analyzed for several exponential and logistic models in [3] by F.A. Milner. It has been shown that the nonreproductive group can indeed alter the population trend and may even make an exponentially increasing population stagnate or decline. A similar result holds for logistic models. D. Maxin and F.A. Milner extended these models in [4] to incorporate a sexually transmitted disease without recovery that does not increase mortality. It has been shown that the abstained groups have the ability to induce a stable disease free equilibrium in an endemic situation. This is quite different from quarantine since the sexually isolated individuals do not reproduce and by this, the number of susceptibles decreases since no vertical transmission is assumed.

In this paper we extend the logistic model from [4] to include vertical transmission. The paper is structured as follows: In the next section we introduce the model and analyze the extinction and the disease free equilibrium and correlate these results with the ones obtained in [4]. We then compute a threshold condition on the non-reproductive rates that describes how the isolated class induces a disease free equilibrium in an endemic situation caused by vertical transmission. We finalize with a reduced model that assumes total vertical transmission when all newborn from infected people are infected at birth. While this is not realistic for the known diseases so far, the stability condition of the interior equilibrium suggests that, contrary to what is expected, a higher isolation rate of infected leads to an endemic equilibrium. This proves again that, though similar in the modeling, the isolation from reproduction acts quite differently than the quarantine.

2. THE LOGISTIC MODEL WITH ABSTAINED GROUPS AND VERTICAL TRANSMISSION

In [4], Maxin and Milner introduced several exponential and logistic models of a sexually transmitted disease that incorporates an abstained class A of people who are isolated from sexual contact. In this paper we will consider the logistic version of their model and we assume that each newborn from an infected individual has a probability ϵ of being healthy at birth. Thus, a fraction of $\beta(1 - \epsilon)I$ are assumed infected at birth. The system becomes

$$(1) \quad \begin{cases} S' &= \beta S + \beta\epsilon I - \lambda SI - (\mu + bP)S - \nu_1 S, \\ I' &= \beta(1 - \epsilon)I + \lambda SI - (\mu + bP)I - \nu_2 I, \\ A' &= \nu_1 S + \nu_2 I - (\mu + bP)A. \end{cases}$$

The Jacobian of this system is

$$J(S, I, A) = \begin{pmatrix} \beta - \lambda I - \mu - bP - bS - \nu_1 & \beta\epsilon - \lambda S - bS & -bS \\ \lambda I - bI & \beta(1 - \epsilon) + \lambda S - \mu - bP - bI - \nu_2 & -bI \\ \nu_1 - bA & \nu_2 - bA & -\mu - bP - bA \end{pmatrix}$$

Notice that with $\epsilon = 1$ the system is identical with the one analyzed in [4]. The model admits an extinction equilibrium $(0, 0, 0)$ and the Jacobian evaluated at this point is

$$J(0, 0, 0) = \begin{pmatrix} \beta - \mu - \nu_1 & \beta\epsilon & 0 \\ 0 & \beta(1 - \epsilon) - \mu - \nu_2 & 0 \\ \nu_1 & \nu_2 & -\mu \end{pmatrix}$$

It follows that the extinction equilibrium is asymptotically stable if

$$\beta - \mu - \nu_1 < 0, \quad \text{and} \quad \beta(1 - \epsilon) - \mu - \nu_2 < 0$$

Let us assume now that $\beta - \mu - \nu_1 > 0$. In this case the model admits a disease free equilibrium (which is identical to the one obtained in [4]):

$$(2) \quad \begin{aligned} S_* &= (K - \frac{\nu_1}{b})(1 - \frac{\nu_1}{\beta}) = (\frac{\beta - \mu - \nu_1}{b})(1 - \frac{\nu_1}{\beta}), \\ I_* &= 0, \\ A_* &= (K - \frac{\nu_1}{b})\frac{\nu_1}{\beta} = (\frac{\beta - \mu - \nu_1}{b})\frac{\nu_1}{\beta}. \end{aligned}$$

Note that $P_* = S_* + A_* = (K - \frac{\nu_1}{b}) = (\frac{\beta - \mu - \nu_1}{b}) > 0$.

The Jacobian of this system is

$$\begin{pmatrix} \beta - \mu - bP_* - bS_* - \nu_1 & \beta\epsilon - \lambda S_* - bS_* & -bS_* \\ 0 & \beta(1 - \epsilon) + \lambda S_* - \mu - bP_* - \nu_2 & 0 \\ \nu_1 - bA_* & \nu_2 - bA_* & -\mu - bP_* - bA_* \end{pmatrix}$$

The eigenvalues are $\beta(1 - \epsilon) + \lambda S_* - \mu - bP_* - \nu_2$ and the eigenvalues of

$$M = \begin{pmatrix} \beta - \mu - bP_* - bS_* - \nu_1 & -bS_* \\ \nu_1 - bA_* & -\mu - bP_* - bA_* \end{pmatrix}$$

We have $\text{tr}(M) = -\mu - 2bP_* < 0$ and $\det(M) > 0$, so these eigenvalues are both negative. Thus we get local asymptotic stability if the following condition holds:

$$\frac{\beta}{K} < \lambda < \frac{\beta\epsilon - \nu_1 + \nu_2}{(1 - \frac{\nu_1}{\beta})(K - \frac{\nu_1}{b})}$$

This condition resembles the similar one obtained in [4] with $\epsilon = 1$.

$$\frac{\beta}{K} < \lambda < \frac{\beta - \nu_1 + \nu_2}{(1 - \frac{\nu_1}{\beta})(K - \frac{\nu_1}{b})}$$

The left hand-side condition represents the endemic situation in the absence of the isolated class A . Thus, if the infection rate λ satisfies the right-hand side bound also the disease free equilibrium becomes stable. This means that the isolated class A , represented by the two isolation rates ν_1 and ν_2 has the ability to induce stability to the disease free equilibrium in an otherwise endemic situation. With the addition of vertical transmission we notice another threshold effect which suggests that the vertical transmission alone can induce an endemic situation even in the case where the abstained class satisfies the condition in [4]. This happens if the infection rate satisfies

$$\frac{\beta\epsilon - \nu_1 + \nu_2}{(1 - \frac{\nu_1}{\beta})(K - \frac{\nu_1}{b})} < \lambda < \frac{\beta - \nu_1 + \nu_2}{(1 - \frac{\nu_1}{\beta})(K - \frac{\nu_1}{b})}$$

To summarize, the vertical transmission reduces the disease-free stability range of λ which is to be expected with the additional infected newborns in the model. A major difference from the model treated in [4] appears when the vertical transmission rate is very high. Although not realistic, for theoretical purposes we will assume the extreme case

$$\epsilon \rightarrow 0$$

which indicates nearly 100% vertical transmission. We treat this case in greater detail in the following section.

3. COMPLETE VERTICAL TRANSMISSION

Replacing $\epsilon = 0$ in (3) we obtain:

$$(3) \quad \begin{cases} S' &= \beta S - \lambda SI - (\mu + bP)S - \nu_1 S, \\ I' &= \beta I + \lambda SI - (\mu + bP)I - \nu_2 I, \\ A' &= \nu_1 S + \nu_2 I - (\mu + bP)A. \end{cases}$$

For simplification, we will let $\bar{\mu}$ be the logistic death rate $\mu + bP$. The system, in this form, allows us to explicitly compute the interior equilibrium (a non-trivial task if $\epsilon \neq 0$):

$$(4) \quad \begin{aligned} S^* &= \frac{\bar{\mu} + \nu_2 - \beta}{\lambda}, \\ I^* &= \frac{\beta - \bar{\mu} - \nu_1}{\lambda}, \\ A^* &= \frac{(\nu_2 - \nu_1)(\beta - \bar{\mu})}{\lambda \bar{\mu}}. \end{aligned}$$

with the condition that $\nu_1 < \beta - \bar{\mu} < \nu_2$.

Adding the equations for S^* , I^* , and A^* together gives us

$$P^* = \frac{(\nu_2 - \nu_1)\beta}{\lambda \bar{\mu}}$$

Biologically, this translates to a requirement that the reproductive number of the susceptibles must be greater than one, while the reproductive number of the infected population must be less than one. Too many infected reproductions or too few susceptible reproductions will cause one of the populations to go extinct.

The Jacobian is:
$$\begin{pmatrix} -bS & -\lambda S - bS & -bS \\ \lambda I - bI & -bI & -bI \\ \nu_1 - bA & \nu_2 - bA & -\mu^* - bA \end{pmatrix}$$

If we denote the characteristic equation by $x^3 + p_1x^2 + p_2x + p_3 = 0$, then:

$$\begin{aligned} p_1 &= \bar{\mu} + bP^* \\ p_2 &= \lambda^2 I^* S^* + b\nu_1 S^* + b\nu_2 I^* + b\mu(S^* + I^*) \\ p_3 &= \lambda S^* I^* (b\nu_2 - b\nu_1 + \lambda\mu + \lambda bA^*) \end{aligned}$$

Clearly $p_1 > 0$, $p_2 > 0$ and $p_3 > 0$ since $\nu_2 > \nu_1$. We also have $p_1 p_2 > p_3$ since

$$p_1 p_2 - p_3 = \frac{b\beta(\nu_2 - \nu_1)[\lambda\mu^2 + b\beta(\nu_2 - \nu_1)]}{\lambda^2 \mu} > 0$$

Hence the Routh-Hurwitz criterion is satisfied and this means the interior equilibrium is always stable whenever it exists. It remains now to interpret the existence condition $\nu_1 < \beta - \bar{\mu} < \mu_2$ in terms of the original parameters. To this end, we solve for $\bar{\mu}$ using the following equation

$$\frac{\bar{\mu} - \mu}{b} = \frac{\beta(\nu_2 - \nu_1)}{\lambda \bar{\mu}}$$

There is a unique positive solution

$$\bar{\mu} = \frac{\mu\lambda + \sqrt{\mu^2\lambda^2 + 4b\beta\lambda(\nu_2 - \nu_1)}}{2\lambda}$$

and the existence condition above becomes

$$2(\beta - \nu_2) - \mu < \frac{1}{\lambda} \sqrt{\mu^2\lambda^2 + 4b\beta\lambda(\nu_2 - \nu_1)} < 2(\beta - \nu_1) - \mu$$

First notice that $\beta > \bar{\mu} + \nu_1$ implies $\beta > \mu + \nu_1 > \frac{\mu}{2} + \nu_1$ so the right-hand side is always positive. Squaring the right-hand side inequality we find that it is equivalent to

$$\lambda > \frac{\nu_2 - \nu_1}{\left(1 - \frac{\nu_1}{\beta}\right) \left(K - \frac{\nu_1}{b}\right)}$$

which is precisely the opposite condition for disease free stability. The left-hand inequality can be trivially satisfied if the left-hand side term is negative, i.e.:

$$\beta < \frac{\mu}{2} + \nu_2$$

Otherwise, if $\beta > \frac{\mu}{2} + \nu_2$, by squaring both sides of the inequality on the left we obtain the following condition

$$\lambda < \frac{\nu_2 - \nu_1}{\left(1 - \frac{\nu_2}{\beta}\right) \left(K - \frac{\nu_2}{b}\right)}$$

This will be the opposite condition of the Healthy Free Equilibrium stability (see below).

For the Healthy Free Equilibrium (HFE) we solve the following system:

$$(5) \quad \begin{cases} 0 = \beta I - \mu^* I - \nu_2 I \\ 0 = \nu_2 I - \mu^* A \end{cases}$$

Which has an equilibrium solution of:

$$(6) \quad \begin{cases} I = (1 - \frac{\nu_2}{\beta})P \\ A = \frac{\nu_2}{\beta}P \\ P = \frac{\beta - \mu - \nu_2}{b} \end{cases}$$

The Jacobian of the system is:

$$\begin{pmatrix} -\lambda I + \nu_2 - \nu_1 & 0 & 0 \\ (\lambda - b)I & -bI & -bI \\ \nu_1 - bA & \nu_2 - bA & -\mu^* - bA \end{pmatrix}$$

It is immediately clear that one of the eigenvalues is negative when $\lambda I > \nu_2 - \nu_1$ which is equivalent to

$$\lambda > \frac{\nu_2 - \nu_1}{\left(1 - \frac{\nu_2}{\beta}\right) \left(K - \frac{\nu_2}{b}\right)}$$

Removing the row and column containing that eigenvalue leaves a 2x2 matrix. The determinant of that matrix is always greater than 0 ($Det = b\beta I$). The Trace is always less than 0:

$$Tr = -\mu^* - bP$$

Thus, the Healthy-free equilibrium with 100% vertical transmission is locally stable with sufficient λ .

We can summarize our results as follows:

If $\beta > \mu + \nu_2$ then there exists a steady state where the entire healthy population vanishes if

$$\lambda > \frac{\nu_2 - \nu_1}{\left(1 - \frac{\nu_2}{\beta}\right) \left(K - \frac{\nu_2}{b}\right)}$$

This is easily explained by the fact that a high infection rate combined with 100% vertical transmission and lack of recovery from the disease eventually causes a full transfer of the susceptibles into the infected class. If

$$\frac{\nu_2 - \nu_1}{\left(1 - \frac{\nu_1}{\beta}\right) \left(K - \frac{\nu_1}{b}\right)} < \lambda < \frac{\nu_2 - \nu_1}{\left(1 - \frac{\nu_2}{\beta}\right) \left(K - \frac{\nu_2}{b}\right)}$$

then the interior endemic equilibrium is stable. It is also stable if

$$\frac{\nu_2 - \nu_1}{\left(1 - \frac{\nu_1}{\beta}\right) \left(K - \frac{\nu_1}{b}\right)} < \lambda, \quad \text{and} \quad \beta < \frac{\mu}{2} + \nu_2$$

Remark 3.1. *The last condition has an interesting consequence. First of all, if $\beta < \frac{\mu}{2} + \nu_2$ then $\beta < \mu + \nu_2$ also which means the Healthy Free Equilibrium does not exist in this case. This means that if ν_2 is big enough then the susceptible class never goes extinct and the interior steady state is stable REGARDLESS of how big the infection rate λ may be. This emphasizes the epidemiological role of the demographic behavior. Since the condition $\beta > \frac{\mu}{2} + \nu_2$ represents an upper bound on the reproductive number of infectious people.*

REFERENCES

- [1] P. F. Verhulst. *Notice sur la Loi que la Population suit dans son Accroissement*, Correspondance mathématique et physique publiée par A. Quételet, Brussels, X, 1838, pp. 113-121
- [2] K. P. Hadeler, R. Waldsttter, and A. Wörz-Busekros. *Models for pair formation in bisexual populations*, J. Math. Biol. 26 (1988), pp. 635-649.
- [3] F. A. Milner. *How may segregation from sexual activity affect population growth?*, Math. Biosc. and Engin. 2 (2005), pp. 579-588.
- [4] D. Maxin, F. A. Milner *Effects of Non-Reproductive Groups on persistent sexually transmitted diseases* Math. Biosc. and Engin. 4 (2007), pp. 505-522.
- [5] K. P. Hadeler and K. Ngoma. *Homogeneous models for sexually transmitted diseases*, Rocky Mt. J. Math. 20 (1990), pp. 967-986.